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# BENDING, EXTENSION, AND TORSION OF NATURALLY TWISTED RODS*

V.L. BERDICHEVSKII and L.A. STAROSEL'SKII

Saint-Venant /1/ established that the spatial problem of linear elasticity theory of the deformation of straight rods with a load-free side surface allows of practically complete investigation: the extension problem is solved exactly (if the boundary layer is ignored), and the bending and torsion problems reduce to Neumann problems for the Laplace equation in the region of the rod cross-section (see /2, 3/). It is shown below that an analogous situation holds for a naturally twisted rod: the spatial problem is successfully reduced to a Neumann-type problem for a certain system of second-order elliptic equations in the cross-section. It is essential that this can be done for an arbitrary value of the rod twist. For zero twist the problem in the section reduces to the Saint-Venant problem. In the case of centrally-symmetric sections, the problem decomposes into two independent problems, on bending and on extension-torsion. Variational principles and certain bilateral estimates of the extension and torsion stiffness are constructed for the latter, and the case of oblong sections is investigated.

The extension-torsion problem for naturally twisted rods was examined earlier in /4/. The difference from this research is discussed in Sect.4.

**1. The undeformed state.** Consider a segment $0 \leqslant x^3 \leqslant l$ located on the $x^3$ axis in a three-dimensional space referred to the Cartesian coordinates $x^i$ (the Latin superscripts run through the values 1, 2, 3). We take a two-dimensional domain $S$ in the $x^3 = 0$ plane and we displace it along the $x^3$ axis while simultaneously rotating it through an angle $\varphi = \omega x^3$, $\omega = \text{const}$ around the $x^3$ axis. A domain $V$ of the type of domains displaced in Fig.1a—1c is noted here. Fig.1a corresponds to the case when the centre of gravity of the cross-section lies on the axis, and Fig.1b when the axis of rotation does not coincide with the line of the centres of gravity of the sections, as holds for turbine blades; Fig.1c is the case when the axis of rotation does not pass through the cross-section. In the later case, bodies of the type of springs are obtained, if $S$ is a circle here, then an ellipse $S'$ is obtained by a plane section through the spring perpendicular to the axial line $\Gamma$, i.e., this will be a spring with an elliptical section in the usual sense. The elliptical domain $S$ corresponds to springs with circular section $S'$.

An elastic body occupying the domain $V$ in the undeformed state is called a naturally twisted rod, while $\omega$ is its twist.

On the axis $x^3 \equiv x$ we introduce a unit orthogonal reference from the vectors $\boldsymbol{\tau}_1(x)$, $\boldsymbol{\tau}_2(x)$ and $\boldsymbol{\tau}$, the vector $\boldsymbol{\tau}$ is directed along the $x$ axis, and the vectors $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$ are rotated during motion along the $x$ axis with velocity $\omega$. The reference $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{\tau}$ is determined by the relationships

$$\frac{d\tau_\alpha{}^i(x)}{dx} = \omega e_{\alpha\cdot}{}^{\beta}\tau_\beta{}^i(x), \quad \tau_\alpha{}^i\tau_{i\beta} = \delta_{\alpha\beta}, \quad \tau_i\tau_\alpha{}^i = 0, \quad \tau^i\tau_i = 1 \tag{1.1}$$

The Greek indices run through the values $1, 2$; $\tau_\alpha{}^i, \tau^i$ are components of the vector $\boldsymbol{\tau}_\alpha$ and $\boldsymbol{\tau}$, respectively, $e_{\alpha\beta}$ are Levi-Civita symbols ($e_{11} = e_{22} = 0$, $e_{12} = -e_{21} = 1$), and summation is over repeated subscripts and superscripts.

The transverse sections $x = \text{const}$ occupy different domains in the variables $x^\alpha, x$ for different values of $x$, and it is convenient to introduce new coordinates, $\xi^\alpha, \xi$ in which the domain $S$ is fixed. They are defined by the equalities

$$\xi = x, \quad x^i = \tau^i\xi + \tau_\alpha{}^i(\xi)\,\xi^\alpha \tag{1.2}$$

The coordinates $\xi^\alpha$ vary in the domain $S$, the coordinate $\xi$ on the segment $[0, l]$. The coordinates $\xi^\alpha$ are accompanying for the domain $S$.

Since $\tau_\alpha{}^\beta(\xi)$ is an orthogonal matrix, $\tau_\alpha{}^3 = 0$, $\tau^\alpha = 0$, $\tau^3 = 1$, then (1.2) can also be rewritten in the form $x^\beta = \tau_\alpha{}^\beta(\xi)\,\xi^\alpha$, $x = \xi$. According to (1.1), the matrix $\tau_\alpha{}^\beta(\xi)$ satisfies the relationship $d\tau_\alpha{}^\beta(\xi)/d\xi = \omega e_{\alpha\cdot}{}^{\gamma}\tau_\gamma{}^\beta$. It can be confirmed that the following formulas hold

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$$\frac{\partial x^i}{\partial \xi} = \tau^i + \omega e_{\alpha}^{\cdot\beta} \xi^\alpha \tau_\beta{}^i, \qquad \frac{\partial x^i}{\partial \xi^\alpha} = \tau_\alpha{}^i \tag{1.3}$$

$$\frac{\partial \xi}{\partial x^i} = \tau_i, \quad \frac{\partial \xi^\alpha}{\partial x^i} = \tau_i{}^\alpha - \omega e_{\beta\cdot}^{\ \alpha} \xi^\beta \tau_i$$

We define the differentiation operation $D$ for any arbitrary scalar function $f(\xi^\alpha, \xi)$ by the formula

$$Df = f_{,\xi} - \omega e_{\beta\cdot}^{\ \alpha} \xi^\beta f_{,\alpha} \tag{1.4}$$

The dot before the superscript $\alpha$ denotes differentiation with respect to $\xi^\alpha$ and the dot before the $\xi$ differentiation with respect to $\xi$. The operator $D$ has the following meaning:

$$Df = \partial f / \partial x \,|_{x^\alpha = \mathrm{const}} \tag{1.5}$$

The last two relationships in (1.3) must be used for the proof.

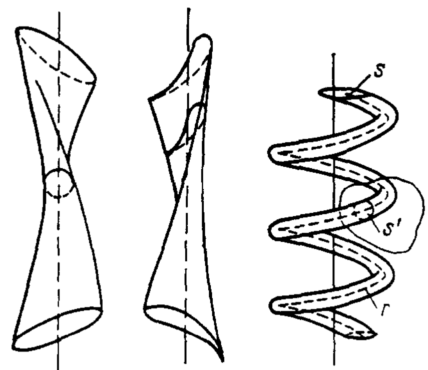
If there is a vector with components $f^\alpha$ in the accompanying coordinates, then we define the operator $D$ by the equality

$$(Df^\alpha)\,\boldsymbol{\tau}_\alpha = \left.\frac{\partial (f^\alpha \boldsymbol{\tau}_\alpha)}{\partial x}\right|_{x^\alpha = \mathrm{const}} = (f^\alpha_{\cdot,\xi} - \omega e_{\beta\cdot}^{\ \sigma} \xi^\beta f^\alpha_{,\sigma} + f^\sigma \omega e_{\sigma\cdot}^{\ \alpha})\,\boldsymbol{\tau}_\alpha \tag{1.6}$$

The definition of the operator $D$ for a second-rank tensor is analogously

$$(Df^{\alpha\beta})\,\boldsymbol{\tau}_\alpha \boldsymbol{\tau}_\beta \equiv \frac{\partial}{\partial x} (f^{\alpha\beta} \boldsymbol{\tau}_\alpha \boldsymbol{\tau}_\beta)\,|_{x^\alpha = \mathrm{const}} =$$

$$(f^{\alpha\beta}_{\cdot,\xi} - \omega e_{\sigma\cdot}^{\ \gamma} \xi^\sigma f^{\alpha\beta}_{,\gamma} + \omega e_{\sigma\cdot}^{\ \alpha} f^{\sigma\beta} + \omega e_{\sigma\cdot}^{\ \beta} f^{\alpha\sigma})\,\boldsymbol{\tau}_\alpha \boldsymbol{\tau}_\beta$$




a b c

Fig.1

The operator $D$ obviously possesses the properties of a covariant differentiation operator

$$D(fg) = (Df)\,g + f\,(Dg), \quad D(f^\alpha g_\alpha) = (Df^\alpha)\,g_\alpha + f^\alpha (Dg_\alpha)$$

**2. Equations of the spatial problem of elasticity theory in accompanying coordinates.** Reduction of the spatial problem of ealsticity theory to a certain two-dimensional problem is based on the selection of projections of the displacements $w^i$ on the vectors $\boldsymbol{\tau}_\alpha, \boldsymbol{\tau}$: $w_\alpha = \tau_\alpha{}^i(x)\, w_i$, $w = \tau^i w_i$, as the desired functions. The quantities $w_\alpha, w$ are sought as functions of $\xi^\alpha, \xi$. We emphasize that $w_\alpha$ and $w$ are not components of the displacement in the accompanying coordinate system $\xi^\alpha, \xi$, since the vectors $\boldsymbol{\tau}_\alpha$ and $\boldsymbol{\tau}$ coincide with the basis vectors of the accompanying coordinate system only on the axis of rotation, while they are different off the axis. In a certain sense such a selection of the desired functions is analogous to the selection of the desired functions in the problem of fluid flow around a body in which the velocity components relative to an inertial reference system are considered as functions of strange coordinates, the coordinates of the inertial system coupled rigidly to the body.

We introduce the system of equations of spatial elasticity theory in which all the functions are assumed to depend on $\xi^\alpha, \xi$

$$\sigma^{\alpha\beta}_{,\beta} + D\sigma^{\alpha 3} = 0, \quad \sigma^{\alpha 3}_{,\alpha} + D\sigma^{33} = 0 \tag{2.1}$$

$$\sigma^{\alpha\beta} = \lambda (\varepsilon_\gamma{}^\gamma + \varepsilon_{33})\,\delta^{\alpha\beta} + 2\mu\varepsilon^{\alpha\beta}, \quad \sigma^{\alpha 3} = 2\mu\varepsilon^{\alpha 3} \tag{2.2}$$

$$\sigma^{33} = \lambda\varepsilon_\gamma{}^\gamma + (\lambda + 2\mu)\,\varepsilon_{33}$$

$$2\varepsilon_{\alpha\beta} = w_{\alpha,\beta} + w_{\beta,\alpha}, \quad 2\varepsilon_{\alpha 3} = w_{,\alpha} + Dw_\alpha, \quad \varepsilon_{33} = Dw \tag{2.3}$$

$$(\sigma^{\alpha\beta} - \omega e_{\sigma\cdot}^{\ \beta} \xi^\sigma \sigma^{\alpha 3})\,\nu_\beta = 0, \quad (\sigma^{\alpha 3} - \omega e_{\sigma\cdot}^{\ \alpha} \xi^\sigma \sigma^{33})\,\nu_\alpha = 0 \quad \text{on } \partial S \tag{2.4}$$

($\nu_\alpha$ are components of the normal to the boundary $\partial S$ of the domain $S$). It is seen the difference between the system of Eqs. (2.1)–(2.4) and the elasticity theory equations in Cartesian coordinates is the replacement of the operator $\partial/\partial x$ by the operator $D$ and a certain complication of the boundary conditions in $\partial S$.

**3. Equations for the integral characteristics.** We define the vectors of the transverse force $Q_\alpha$ and bending moments $M_\alpha$, the torque $M$ and the axial tensile force $T$ by the formulas

$$Q_\alpha = \langle\!\langle \sigma_{\alpha 3} \rangle\!\rangle, \quad M_\alpha = \langle\!\langle \sigma_{33} \xi_\alpha \rangle\!\rangle, \quad M = \langle\!\langle e_{\alpha\beta} \sigma^{\alpha 3} \xi^\beta \rangle\!\rangle, \quad T = \langle\!\langle \sigma_{33} \rangle\!\rangle \tag{3.1}$$

(the «·» is the integral over the section $S$). We will derive equations that the integral characteristics of the state of stress $Q_\alpha, M_\alpha, M, T$ satisfy. To do this, we rewrite the equilibrium Eq.(2.1) by taking account of the definition (1.4), (1.6) of the operator $D$ in the form

$$(\sigma^{\alpha\beta} - \omega e_{\sigma}^{\cdot\beta}\xi^\sigma\sigma^{\alpha 3})_{,\beta} + \sigma^{\alpha 3}_{,\xi} + \omega e_{\beta}^{\cdot\alpha}\sigma^{\beta 3} = 0 \tag{3.2}$$
$$(\sigma^{\alpha 3} - \omega e_{\sigma}^{\cdot\alpha}\xi^\sigma\sigma^{33})_{,\alpha} + \sigma^{33}_{,\xi} = 0$$

The identities

$$(\omega e_{\sigma}^{\cdot\beta}\xi^\sigma\sigma^{\alpha 3})_{,\beta} \equiv \omega e_{\sigma}^{\cdot\beta}\xi^\sigma\sigma^{\alpha 3}_{,\beta}$$
$$(\omega e_{\sigma}^{\cdot\alpha}\xi^\sigma\sigma^{33})_{,\alpha} \equiv \omega e_{\sigma}^{\cdot\alpha}\xi^\sigma\sigma^{33}_{,\alpha}$$

are used here.

Integrating (3.2) over $S$ and using the boundary conditions (2.4), we have

$$DQ_\alpha = 0, \; DT = 0 \tag{3.3}$$

We note that for functions independent of $\xi^\alpha$, that are integral characteristics, the operator $D$ applied to a scalar will agree, according to (1.4), with the derivative with respect to $\xi$ and $DT \equiv \partial T/\partial\xi$ while the operator $D$ applied to a two-dimensional vector will, according to (1.6), have the form $DQ_\alpha \equiv \partial Q_\alpha/\partial\xi + \omega e_{\alpha}^{\cdot\sigma}Q_\sigma$.

Now we multiply the second equation in (3.2) by $\xi^\alpha$ and we integrate the result over $S$. Integrating by parts and using the boundary conditions (2.4), we obtain the equation

$$DM_\alpha = Q_\alpha \tag{3.4}$$

Convoluting the first equation of (3.2) with $e_{\alpha\sigma}\xi^\sigma$ and integrating over $S$, we find as before

$$DM = 0 \tag{3.5}$$

Eqs.(3.3)–(3.5) are the equilibrium equations of one-dimensional rod theory and mean that the tensile force and torque are constant along the axis, the transverse force is constant in the sense that $\partial(Q^\alpha\tau_\alpha)/\partial\xi = 0$, while the bending moments are related to the transverse forces by means of (3.4).

**4. Problem on extension-torsion.** The possibility of an independent investigation of the extension-torsion problem is related to the symmetry properties of the transverse section. Later in Sect.10 it will follow from the above that for rods with centrally symmetric transverse section (the section contains the point $-\xi^\alpha$ in addition to each point $\xi^\alpha$), the bending problem is separated from the extension-torsion problem. We shall consider the section centrally symmetric and we consider the deformation of a rod for which the displacements have the form

$$w = \gamma\xi + w'(\xi^\gamma), \quad u_\alpha = \Omega\xi e_{\alpha\beta}\xi^\beta + u_\alpha'(\xi^\gamma) \tag{4.1}$$

where $\gamma, \Omega$ are certain constants, while $w'$ and $u_\alpha'$ are functions of the coordinates of the transverse section which will be sought later. Since the functions $w, u_\alpha$ are not encountered in subsequent formulas, the primes are omitted on $w', u_\alpha'$.

Evaluating the deformation, we have

$$2\varepsilon_{\alpha\beta} = u_{\alpha,\beta} - u_{\beta,\alpha}, \quad 2\varepsilon_{\alpha 3} = \Omega e_{\alpha\beta}\xi^\beta + w_{,\alpha} - \omega e_{\alpha}^{\cdot\beta}u_\beta - \tag{4.2}$$
$$\omega e_{\beta}^{\cdot\sigma}\xi^\beta u_{\alpha,\sigma}, \quad \varepsilon_{33} = \gamma - \omega e_{\beta}^{\cdot\alpha}\xi^\beta u_{,\alpha}$$

It is seen that the deformation is independent of $\xi$, consequently the stresses will also be independent of $\xi$. Therefore, the system of equations will consist of the equilibrium equations

$$(\sigma^{\alpha\beta} - \omega e_{\sigma}^{\cdot\beta}\xi^\sigma\sigma^{\alpha 3})_{,\beta} + \omega e_{\beta}^{\cdot\alpha}\sigma^{\beta 3} = 0 \tag{4.3}$$
$$(\sigma^{\alpha 3} - \omega e_{\sigma}^{\cdot\alpha}\xi^\sigma\sigma^{33})_{,\alpha} = 0$$

the equations of state (2.2), the kinematic relationships (4.2), and the boundary conditions (2.4). The conditions for the integrability of problem (4.3), (2.4) are satisfied because of the evident equalities $Q^\alpha = \langle\!\langle\sigma^{\alpha 3}\rangle\!\rangle = 0$. If it is rewritten in terms of the functions $w, u_\alpha$ a system of three-second-order equations is obtained in the domain $S$ with Neumann-type boundary conditions. For zero twist $\omega$ the equations reduce to the appropriate Saint-Venant type equations.

*Remark.* The problem of extension-torsion of a naturally twisted rod was considered in /4/. In this paper there are basic considerations about the selection of dependences of the projections of the displacements on the Cartesian axes on the accompanying coordinates as desired functions. The difference from what is elucidated above is in the selection of the displacement modes. They are taken in the form (compare with (4.1))

$$w = \gamma\xi + \psi(\xi^\alpha), \quad u_\alpha = \Omega\xi e_{\beta\alpha}\xi^\beta + f(\xi^\gamma)_{,\alpha} - \xi_\alpha\omega(\xi^\gamma)$$

and comprise equations to determine the desired functions $\psi, f, \omega$. In this connection, equations of higher order were obtained than those which can be obtained by starting from (4.1) and (4.3).

Therefore, the result of Sect.4 can be interpreted as proof of the possibility of reducing the order of the system of governing Eqs. /4/.

**5. Variational principle.** A system of equations of the extension-torsion problem for a rod can be obtained as a system of Euler equations for the functional

$$I = \langle\langle U \rangle\rangle \tag{5.1}$$
$$2U = \lambda(\varepsilon_\alpha{}^\alpha + \varepsilon_{33})^2 + 2\mu(\varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta} + 2\varepsilon_{\alpha3}\varepsilon^{\alpha3} + \varepsilon_{33}{}^2)$$

where the deformation components are expressed in terms of the functions $w, w_\alpha$ by means of (4.2) and the minimum is sought over all functions $w, w_\alpha$ for fixed parameters $\gamma$ and $\Omega$. This assertion is confirmed by direct calculation.

If solid motions are eliminated by the conditions

$$\langle\langle w \rangle\rangle = 0,\ \langle\langle e_{\alpha\beta} w^\alpha \xi^\beta \rangle\rangle = 0$$

then the functional (5.1) becomes strictly convex and it can be asserted that the solution of the problem exists and is unique.

The lower boundary of the functional $\underline{I}$ is a quadratic form in the parameters $\gamma$ and $\Omega$

$$\underline{I} = {}^1/_2\,(E\gamma^2 + 2B\gamma\Omega + C\Omega^2) \tag{5.2}$$

The following formulas hold

$$T = \partial\underline{I}/\partial\gamma = E\gamma + B\Omega,\ M = \partial\underline{I}/\partial\Omega = B\gamma + C\Omega \tag{5.3}$$

Consequently, the coefficient $E$ has the meaning of an effective Young's modulus, the coefficient $C$ is the rod torsion stiffness, and the coefficient $B$ describes the cross effect of the origination of extension forces during torsion and of a torque during tension.

Let us prove (5.3). We give arbitrary increments $\delta\gamma, \delta\Omega$ to the parameters $\gamma$ and $\Omega$. The minimizing functions $w^\circ$ and $w_\alpha{}^\circ$ also receive certain increments $\delta w^\circ, \delta w_\alpha{}^\circ$ here. The change in the minimum value of the functional $\delta\underline{I}$ will equal

$$\delta\underline{I} = \langle\langle \sigma^{\alpha3} e_{\alpha\beta}\xi^\beta \delta\Omega + \sigma^{33}\delta\gamma \rangle\rangle = M\delta\Omega + T\delta\gamma \tag{5.4}$$

The terms containing $\delta w^\circ$ and $\delta w_\alpha{}^\circ$ are cancelled by virtue of the Euler equations of the functional $\underline{I}$. And (5.3) follows from (5.4) and (5.2).

**6. Dual variational principle and stress functions.** The dual variational principle is formulated as follows:

$$\underline{I} = \sup\left[\langle\langle \sigma^{\alpha3} e_{\alpha\beta}\xi^\beta \rangle\rangle\,\Omega + \langle\langle \sigma^{33} \rangle\rangle\,\gamma - \langle\langle U^* \rangle\rangle\right] \tag{6.1}$$
$$U^* = [(1+\nu)(\sigma^{\alpha\beta}\sigma_{\alpha\beta} + 2\sigma^{\alpha3}\sigma_{\alpha3} + (\sigma^{33})^2) - \nu(\sigma_\alpha{}^\alpha + \sigma^{33})^2]/(2E)$$

Here the upper bound is sought over all functions $\sigma^{\alpha\beta}, \sigma^{\alpha3}, \sigma^{33}$ satisfying the equilibrium Eqs.(4.3) and the boundary conditions (2.4). It is constructed according to the general rule (/5/, Sect.3, Ch.II).

We construct the general solution of (4.3) by introducing the appropriate stress functions. The second equation of (4.3) means that a function $\chi$ exists such that the following equality holds:

$$\sigma^{\alpha3} - \omega e_{\sigma\cdot}^{\cdot\alpha}\xi^\sigma\sigma^{33} = e^{\alpha\beta}\chi_{,\beta} \tag{6.2}$$

The quantity $\sigma^{33}$ does not occur in the first equation in (4.3), hence, $\sigma^{33}$ and $\chi$ can be regarded as arbitrary functions, and Eqs.(6.2) here express the quantity $\sigma^{\alpha3}$ in terms of $\sigma^{33}$ and $\chi$.

We examine the expression $\omega e_{\beta\cdot}^{\cdot\alpha}\sigma^{\beta3}$ in the first equation of (4.3). According to (6.2) it can be rewritten in the form

$$\omega e_{\beta\cdot}^{\cdot\alpha}\sigma^{\beta3} = \omega\chi_{,}{}^{\alpha} - \omega^2\xi^\alpha\sigma^{33} \tag{6.3}$$

Without loss of generality, the function $\sigma^{33}$ can be represented in the form

$$\sigma^{33} = 3\psi + \xi^\nu\psi_{,\nu} \tag{6.4}$$

Indeed, (6.4) in the $r, \theta$ polar coordinate system reduces to the ordinary differential equation $3\psi + r\partial\psi/\partial r = \sigma^{33}$, which enables $\psi$ to be evaluated in terms of $\sigma^{33}$.

Direct substitution verifies that after (6.4) has been substituted, (6.3) reduces to the equation

$$\omega e_{\beta\cdot}^{\cdot\alpha}\sigma^{\beta3} = (\omega\chi\delta^{\alpha\beta} - \omega^2\psi\xi^\alpha\xi^\beta)_{,\beta} \tag{6.5}$$

By virtue of (6.5) the first equilibrium equation in (4.3) takes the form

$$(\sigma^{\alpha\beta} - \omega e_{\sigma\cdot}^{\cdot\beta}\xi^\sigma\sigma^{\alpha3} + \omega\chi\delta^{\alpha\beta} - \omega^2\psi\xi^\alpha\xi^\beta)_{,\beta} = 0 \tag{6.6}$$

which means that functions $\varphi^\alpha$ exist such that

$$\sigma^{\alpha\beta} - \omega e_{\sigma\cdot}^{\cdot\beta}\xi^\sigma\sigma^{\alpha3} + \omega\chi\delta^{\alpha\beta} - \omega^2\psi\xi^\alpha\xi^\beta = e^{\beta\lambda}\varphi^\alpha_{,\lambda} \tag{6.7}$$

It remains to satisfy the symmetry condition of the tensor $\sigma^{\alpha\beta}$ in $\alpha, \beta$ which is obtained by convolution of Eq.(6.7) with $e^{\alpha\beta}$

$$\omega\xi_\alpha\sigma^{\alpha3} = \varphi^\alpha_{,\alpha} = \omega\xi_\alpha e^{\alpha\beta}\chi_{,\beta} = (\omega e^{\alpha\beta}\xi_\alpha\chi)_{,\beta} \tag{6.8}$$

The relationship (6.2) is used here. Formula (6.8) means that the vector $\varphi^\alpha + \omega e^{\alpha\beta}\xi_\beta\chi$ has zero divergence, i.e., a function $\varphi$ exists such that

$$\varphi^\alpha + \omega e^{\alpha\beta}\xi_\beta\chi = e^{\alpha\beta}\varphi_{,\beta} \tag{6.9}$$

Thus, the general solution of the equilibrium equation is expressed in terms of three functions $\varphi, \psi$ and $\chi$ by means of the formulas

$$\begin{aligned} \sigma^{\alpha\beta} &= e^{\alpha\mu}e^{\beta\lambda}\varphi_{,\mu\lambda} + \omega\,[e_{\sigma\cdot}^{\cdot\beta}\xi^\sigma e^{\alpha\lambda}\chi_{,\lambda} + e^{\sigma\alpha}\xi_\sigma e^{\beta\lambda}\chi_{,\lambda} - 2\delta^{\alpha\beta}\chi] + \\ &\quad \omega^2\,[e_{\lambda\cdot}^{\cdot\beta}e_{\mu\cdot}^{\cdot\alpha}\xi^\lambda\xi^\mu\,(3\psi + \xi^\nu\psi_{,\nu}) + \psi\xi^\alpha\xi^\beta] \\ \sigma^{\alpha3} &= e^{\alpha\beta}\chi_{,\beta} + \omega e_{\sigma\cdot}^{\cdot\alpha}\xi^\sigma\,(3\psi + \xi^\nu\psi_{,\nu}) \\ \sigma^{33} &= 3\psi + \xi^\nu\psi_{,\nu} \end{aligned} \tag{6.10}$$

The function $\varphi$ is the analogue of the Airy function while the function $\chi$ is the torsion function.

Let us determine what arbitrariness remains in the stress functions for a fixed state of stress. It is evidently sufficient to consider the case of a zero state of stress. Setting $\sigma^{\alpha\beta} = \sigma^{\alpha3} = \sigma^{33} = 0$, we obtain

$$\chi_{,\alpha} = 0, \quad 3\psi + \xi^\nu\psi_{,\nu} = 0, \quad e^{\alpha\mu}e^{\beta\lambda}\varphi_{,\mu\lambda} - 2\omega\chi\delta^{\alpha\beta} + \omega^2\psi\xi^\alpha\xi^\beta = 0$$

It follows from the first equation that $\chi = c = \text{const}$. The second equation is integrated in polar coordinates: $\psi = a(\theta)\,r^{-3}$. We will consider the origin to lie in the domain $S$ and the stress functions under consideration to have no singularities in $S$. Then $\psi \equiv 0$ and the third equation yields $\varphi = \text{const} + c_\alpha\xi^\alpha + \omega c\xi_\alpha\xi^\alpha$, where $c_\alpha$ are constants. Thus, the function $\chi$ is determined, apart from the constant $c$, the function $\varphi$ to the accuracy of a linear function, and the term $\omega c\xi_\alpha\xi^\alpha$ while the function $\psi$ has no arbitrariness.

According to (6.2) and the second boundary condition (2.4), the function $\chi$ is constant on $\partial S$. According to (6.7), (6.9) and the first boundary condition (2.4), the function $\varphi$ satisfies the following equation on the boundary of the domain $S$

$$\frac{d}{ds}\,(e^{\alpha\beta}\varphi_{,\beta} - \omega e^{\alpha\beta}\xi_\beta\chi) = \omega\chi\nu^\alpha - \omega^2\psi\xi^\alpha\xi_\beta\nu^\beta \tag{6.11}$$

Here $s$ is the length of the arc along $\partial S$, and it is assumed that the domain $S$ remains on the left during motion along $\partial S$ in the direction of the tangent vector $\tau^\alpha = d\xi^\alpha/ds$ while the normal vector $\nu^\alpha$ is directed outside $S$ so that $\nu^\alpha = e^{\alpha}_{\cdot\beta}\tau^\beta,\ \tau^\lambda = e^{\beta\lambda}\nu_\beta$.

In a simply-connected domain the function $\chi$ can be assumed to be zero on $\partial S$, consequently, the relationship (6.11) between $\varphi$ and $\psi$ acquires the form

$$\frac{d}{ds}\,(e^{\alpha\beta}\varphi_{,\beta}) = -\,\omega^2\psi\xi^\alpha\xi_\beta\nu^\beta \tag{6.12}$$

We consider the functional in (6.1) as a functional of the functions $\varphi, \psi$ and $\chi$, which is obtained by substituting (6.10) into (6.1) instead of the stress tensor components. The dual variational principle in the stress functions is formulated as follows: the true state of stress brings the maximum to the functional in (6.1) in the set of functions $\varphi, \psi, \chi$ satisfying the constraint (6.11) and the condition $\chi = \text{const}$ on $\partial S$. For a simply-connected domain they can be replaced by (6.12) and the condition $\chi = 0$ on $\partial S$.

**7. Some estimates.** The roughest estimate of the effective stiffnesses is obtained if we set $w = w_\alpha = 0$. We have from (5.1), (5.2) and (4.2)

$$\tfrac{1}{2}\,(E\gamma^2 + 2B\gamma\Omega + C\Omega^2) \leqslant \tfrac{1}{2}\,[(\lambda + 2\mu)\,|\,S\,|\,\gamma^2 + \mu I_\alpha{}^\alpha\Omega^2]$$

Here the inertia tensor of the transverse section is denoted by $I^{\alpha\beta} = \langle\!\langle \xi^\alpha\xi^\beta \rangle\!\rangle$, and $|\,S\,|$ is the area of the domain $S$. A more exact estimate is obtained if we set $w_\alpha = a\xi_\alpha$, $w = {}^1/_2 b_{\alpha\beta}\xi^\alpha\xi^\beta$ and the unknown coefficients a and $b_{\alpha\beta}$ are sought from the condition of the minimum of the functional (5.1). Then we find from (5.1), (5.2) and (4.2)

$$\begin{aligned} &{}^1/_2\,(E\gamma^2 + 2B\gamma\Omega + C\Omega^2) \leqslant {}^1/_2\,(E^+\gamma^2 + 2B^+\gamma\Omega + C^+\Omega^2) \\ &E^+ = E\,[\,|\,S\,| - 2\,(1 + \nu)\,\omega^2\,(I_1 - I_2)^2/\Delta] \\ &B^+ = -E\omega\,(I_1 - I_2)^2/\Delta \\ &C^+ = \mu\,[I_\alpha{}^\alpha - (I_1 - I_2)^2/\Delta],\ I_1 = I^{11},\ I_2 = I^{22} \\ &\Delta = I_1 + I_2 + \omega^2\,[(2 + \lambda/\mu)\,\langle\!\langle(\xi_1{}^2 - \xi_2{}^2)^2\rangle\!\rangle - \\ &\qquad 4\nu^2\,(I_1 - I_2)^2/((1 - 2\nu)\,|\,S\,|)] \end{aligned} \tag{7.1}$$

If the rod is not twisted ($\omega = 0$), then, as is easy to see, $B = 0$ and the inequality (7.1) reduces to the estimate $E \leqslant E\,|\,S\,|$, $C \leqslant 4\mu I_1 I_2/(I_1 + I_2)$. The former yields the exact value of the longitudinal stiffness for an untwisted rod, while the latter is the Nikolai inequality /6/.

If the twist $\omega$ is not zero, inequality (7.1) reduces to three estimates

$$\hat{E} \leqslant E^+, \ (B^+ - B)^2 \leqslant (C^+ - C)(E^+ - \hat{E}), \ C \leqslant C^+ \tag{7.2}$$

The quantity $\Delta$ in (7.1) is non-negative, hence $E^+$ decreases as the twist increases. Therefore, according to (7.2) the effective Young's modulus of a twisted rod with arbitrary transverse section is less than for an untwisted rod with the same transverse section. In the limit as $\omega \to \infty$ the quantity $E^+$ reduces to the value

$$E \mid S \mid [1 - (1 + \nu)(1 - 2\nu)(\varkappa - 2\nu^2)^{-1}],$$
$$\varkappa = (1 - \nu) \mid S \mid \langle\langle(\xi_1^2 - \xi_2^2)^2\rangle\rangle (I_1 - I_2)^{-2}$$

The second inequality estimates the amplitude of the cross effect. The third is a generalization of the Nikolai inequality to a naturally twisted rod.

We present the values of $E^+, B^+, C^+$ for elliptical sections $(\xi_1^2/a^2 + \xi_2^2/b^2 \leqslant 1)$ with semi-axis ratio $c = b/a$

$$\begin{aligned} E^+ &= E\pi ab\,[1 - \bar{\omega}^2 (1 + \nu)(1 - c^2)^2/(2(1 + c^2 + \bar{\omega}^2\alpha))] \\ B^+ &= -E\omega\pi ba^3 (1 - c^2)^2/(4(1 + c^2 + \bar{\omega}^2\alpha)) \\ C^+ &= \mu\pi ba^3\,[c^2 + \bar{\omega}^2 (1 + c^2)\,\alpha/4]/(1 + c^2 + \bar{\omega}^2\alpha) \\ \bar{\omega} &= a\omega, \ \alpha = [(1 - \nu - \nu^2)(1 + c^4) - {}^2/_3 c^2 (1 - \nu - 3\nu^2)]/(1 - 2\nu) \end{aligned} \tag{7.3}$$

The estimates (7.2) are valid for rods with arbitrary centrally symmetric section. In order to characterize the error of these estimates, we construct a lower bound of the effective characteristics for an elliptical section. We set $\chi = A(\xi_1^2/a^2 + \xi_2^2/b^2 - 1), \varphi = {}^1/_2 (B_1\xi_1^2 + B_2\xi_2^2)$, $A, B_1, B_2$ are constants. Then the boundary condition for $\chi$ is satisfied while the boundary condition for (6.12) takes the form

$$(B_2/a^2 + \omega^2\psi)\,\xi_1 = 0, \ (B_1/b^2 + \omega^2\psi)\,\xi_2 = 0 \tag{7.4}$$

We take the simplest expression for the function $\psi$, thus $\psi = B/\omega^2$. The condition (7.4) will be satisfied if we set $B_1 = -Bb^2, B_2 = -Ba^2$. We select the constants $A$ and $B$ from the condition of maximum of the functional (6.1). We obtain from (6.10) and (6.1)

$$\begin{aligned} &{}^1/_2 (E^-\gamma^2 + 2B^-\gamma\Omega + C^-\Omega^2) \leqslant {}^1/_2 (\hat{E}\gamma^2 + 2B\gamma\Omega + C\Omega^2) \\ &E^- = E_0, \ E_0 = E \mid S \mid [1 - \bar{\omega}^2 (1 + \nu)(1 - c^2)^2/(2(1 + c^2))] \\ &B^- = -E\omega\pi ba^3 (1 - c^2)^2/(4(1 + c^2)) \\ &C^- = \mu\,\frac{\pi ba^3c^2}{1 + c^2}\{1 + \bar{\omega}^2 [3(1 + \nu)^2 (1 - c^2)^4 - 8(1 + \nu)c^2(1 + c^4) - \\ &\quad 8(38 + 22\nu)c^4][48(1 + \nu)c^2(1 + c^2)]^{-1}\} \end{aligned} \tag{7.5}$$

The expressions for $E^-, B^-, C^-$ are presented for the case of small twists $\bar{\omega}$ (the terms containing $\bar{\omega}^3$ and higher powers of $\bar{\omega}$ are omitted) and $c \lessapprox 1$. It is seen from a comparison of (7.3) and (7.5) that for small $\bar{\omega}$ the upper and lower bounds of the effective Young's modulus converge for rods of elliptical section and define its exact value: $\hat{E} = E^+ = E^- = E_0$. The difference between the effective Young's modulus of the twisted and untwisted rods is characterized by the following numbers: for $\bar{\omega} = 0.5, \ c = 0.1, \ \nu = 0.3$ the effective Young's modulus for the twisted rod is 16% less.

The expressions for $C^+$ and $C^-$ do not agree in the case of small twists. This indicates that the selected allowable fields are good enough to approximate the state of stress caused by extension and unsatisfactory to describe torsion.

The cross coefficients in the lower and upper estimates $B^-$ and $B^+$ agree in the limit of small twists; however, because of the difference between $C^-$ and $C^+$ this does not enable us to give an exact value of $B$.

The case when the problem contains small parameters, small twist or small thickness of the transverse section (elongated section), allows of a full asymptotic investigation.

We consider here the case of rods with elongated sections, which is often encountered in engineering applications (see /7/).

**8. Elongated sections.** We consider a rod with cross-sections for which the coordinate axes are the axes of symmetry. We consider part of the boundary $\partial S$ with $\xi_2 \geqslant 0$ to be projected single-valuedly on the axis $\xi_1$ and its equation to be written in the form $\xi_2 = h(\xi_1), |\xi_1| \leqslant a$. We let $b$ denote the maximum value of $h$. By assumption $h$ is an even function of $\xi_1$, that varies slightly at distances of the order of $b$ and $b \ll a$.

We perform an asymptotic analysis of the functional (5.1). We rewrite the elastic energy density in the form

$$\begin{aligned} 2U &= E\varepsilon_{33}^2 + E^{\alpha\beta\gamma\delta}(\varepsilon_{\alpha\beta} + \nu\varepsilon_{33}\delta_{\alpha\beta})(\varepsilon_{\gamma\delta} + \nu\varepsilon_{33}\delta_{\gamma\delta}) + 4\mu\varepsilon_{\alpha3}\varepsilon^{\alpha3} \\ E^{\alpha\beta\gamma\delta} &= \lambda\delta^{\alpha\beta}\delta^{\gamma\delta} + \mu(\delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\beta\gamma}\delta^{\alpha\delta}) \end{aligned} \tag{8.1}$$

The strain is replaced in (8.1) by their expression in terms of the displacement (4.2), and new desired functions $v_\alpha$ are introduced in place of $w_\alpha$ by means of the formula $w_\alpha = -\nu\gamma\xi_\alpha + v_\alpha$. We obtain

$$2U = E(\gamma - \omega\xi_1 w_{,2} + \underline{\omega\xi_2 w_{,1}})^2 + \frac{\lambda^2}{\lambda+\mu}\omega^2(\xi_1 w_{,2} - \underline{\xi_2 w_{,1}})^2 + \lambda(v_{1,1} + v_{2,2})^2 + 2\mu\left(\underline{v_{1,1}^2} + v_{2,2}^2 + \frac{1}{2}v_{1,2}^2 + \underline{\frac{1}{2}v_{2,1}^2 + v_{1,2}v_{2,1}}\right) - 2\lambda\omega(v_{1,1} + v_{2,2})(\xi_1 w_{,2} - \underline{\xi_2 w_{,1}}) + \mu(w_{,1} + \xi_2\Omega - \omega v_2 - \omega\xi_1 v_{1,2} + \underline{\omega\xi_2 v_{1,1}})^2 + \mu(w_{,2} - \xi_1\Omega + \omega v_1 - w\xi_1 v_{2,2} + \underline{\omega\xi_2 v_{2,1}})^2 \tag{8.2}$$

It is seen from this expression that the minimizing functions $w$ and $v_2$ are odd functions of $\xi_2$ (and therefore equal zero for $\xi_2 = 0$), while $v_1$ is an even function of $\xi_2$. The proof is carried out exactly as the proof of the analogous assertion in /8/.

We shall assume that the twist not small and $a\omega \sim 1$. We will find the orders of the desired functions. Poisson's ratio is obviously not essential in determining the orders of the functions; consequently we assume $\lambda = 0$.

If $\gamma = \Omega = 0$, then the minimizing functions of the functional $\langle U\rangle$ also equal zero. The difference of the solutions from zero is generated by the cross terms between $\gamma, \Omega$ and the desired functions. Consequently, a rough estimation of the orders reduces to retaining just the principal terms in the desired functions and the practical cross terms in the functional. In extracting the principal terms we consider the derivatives of the desired functions with respect to $\xi_2$ to be very much greater than the derivatives with respect to $\xi_1$. For this reason, all the underlined terms in (8.2) can be discarded as small compared with the preceding terms. Keeping the principal terms in $U$ we obtain

$$2U_0 = -2E\omega\gamma\xi_1 w_{,2} + 2\mu\left(v_{2,2}^2 + \frac{1}{2}v_{1,2}^2\right) + \mu(-2\omega\xi_2\Omega v_2 - 2\xi_2\omega\Omega\xi_1 v_{1,2}) + \mu(w_{,2}^2 - 2w_{,2}\xi_1\Omega - 2\xi_1\omega\Omega v_1 + 2\omega\Omega\xi_1^2 v_{2,2}) \tag{8.3}$$

Here all the cross terms between the desired functions are omitted since a rough estimate is made of the orders; moreover the terms $E\omega^2\xi_1^2 w_{,2}^2$ and $\mu\omega^2\xi_1^2 v_{1,2}^2$, which have the same form as the retained terms $\mu w_{,2}^2$ and $\mu v_{1,2}^2$, are omitted, and do not influence the order of the desired functions. Minimizing the simplified functional $\langle U_0\rangle$ with respect to $w$ and $v_2$, we find (it is here necessary to take into account that $w = v_2 = 0$ for $\xi_2 = 0$)

$$w \sim ab(\Omega + \omega\gamma), \; v_2 \sim a^2 b\omega\Omega \tag{8.4}$$

However, the problem of the minimum of the functional $\langle U_0\rangle$ with respect to $v_1$ turns out to be incorrect: shifts in $v_1$ by a constant can carry the value of the functional to $-\infty$. Consequently, the intitial assumption on the nature of the behaviour of $v_1$ must be altered somewhat. We extract the constant component in $\xi_2$ from the function $v_1$. Without loss of generality, we can write $v_1 = u(\xi_1) + v(\xi_1, \xi_2)$, where the function $v$ satisfies the condition $\langle v\rangle = 0$ ($\langle\cdot\rangle$ is the integral with respect to $\xi_2$ within the limits $[-b, b]$). We consider that $|v_{,2}| \gg v_1$, and by defining $u(\xi_1)$, we find the order of $v$. The principal term in $v$ will be $\mu v_{,2}^2$, the principal cross term between $v$ and $\Omega$ is $-2\mu\omega\Omega\xi_1\xi_2 v_{,2}$, the cross term $2\mu\omega\Omega\mu_1 v$ equals zero by virtue of the condition $\langle v\rangle = 0$. The problem of finding $v$ is now correct and shows that $v \sim ab^2\omega\Omega$.

For $w = v_\alpha = 0$ the value of the functional has the order $\mu(\gamma^2 + a^2\Omega^2)ab$. It is clear that this is the greatest possible order of any of the energy terms. We write $u_{,1}$ for the greatest possible order $\gamma + a\Omega$, and u for the order $(\gamma + a\Omega)a$ and we extract the principal terms in the energy (8.2) by using the estimates cited. Keeping only the principal terms we obtain

$$2U = E(\gamma - \omega\xi_1 w_{,2})^2 + \frac{\lambda^2}{\lambda+\mu}\omega^2\xi_1^2 w_{,2}^2 + \lambda(u_{,1} + v_{2,2})^2 + 2\mu(u_{,1}^2 + v_{2,2}^2) - 2\lambda\omega(u_{,1} + v_{2,2})\xi_1 w_{,2} + \mu(w_{,2} - \xi_1\Omega + \omega u - \omega\xi_1 v_{2,2})^2 \tag{8.5}$$

All the terms in the energy are of the order $\mu(\gamma + a\Omega)^2$ (for $a\omega \sim 1$), and all the discarded terms are higher-order infinitesimals. Therefore, (8.5) yields the value of the energy to a first approximation.

The energy depends on the functions $w$ and $v_2$ only in terms of the derivatives $w_{,2}$ and $v_{2,2}$. Consequently, $w_{,2}$ and $w_{2,2}$ can be selected as new independent desired functions. Then minimizing $\langle U\rangle$ with respect to $w$ and $v_2$ reduces to an algebraic problem of minimizing the quadratic form (8.5) in the arguments $w_{,2}$ and $v_{2,2}$. The minimum is achieved on the functions

$$v_2 = -D^{-1}\left[\mu\omega\Omega\xi_1^2 + \lambda u_{,1} - \mu\omega^2\xi_1 u - \frac{1}{A}(\lambda^2\omega^2\xi_1^2 u_{,1} + (\lambda+\mu)\omega\xi_1^2(\mu\Omega + \omega E\gamma) - \lambda\mu\omega^2\xi_1(u - \xi_1 u_{,1}) - \mu^2\omega^2\xi_1 u\right]\xi_2$$

$$w = A^{-1}[\mu\xi_1\Omega + E\omega\xi_1\gamma + \lambda\omega\xi_1(u_{,1} + v_{2,2}) + \mu\omega\xi_1 v_{2,2} - \mu\omega u]\xi_2$$

$$D = \lambda + 2\mu + [\mu - (\lambda+\mu)^2/A]\omega^2\xi_1^2, \; A = \mu + [E + \lambda^2/(\lambda+\mu)]\omega^2\xi_1^2$$

The energy here becomes a functional of $u$

$$\langle U\rangle = \int_{-a}^{a}(\alpha u_{,1}^2 - 2\beta u_{,1} + \delta u^2 - 2\varepsilon u + 2\eta u u_{,1})h(\xi_1)d\xi_1 + \tag{8.6}$$

$$\int_{-a}^{a}\Big[E\gamma^2+\mu\xi_1^2\Omega-\frac{1}{A}(\mu\Omega+E\omega\gamma)^2\xi_1^2-$$

$$\frac{1}{D}\Big(\mu\omega\xi_1^2\Omega-\frac{\lambda+\mu}{A}\omega\xi_1^2(\mu\Omega+E\omega\gamma)\Big)^2\Big]h(\xi_1)\,d\xi_1$$

$$\alpha=\lambda+2\mu-\frac{\lambda^2}{A}\omega^2\xi_1^2-\frac{\lambda^2}{D}\Big(1-\frac{\lambda+\mu}{A}\omega^2\xi_1^2\Big)^2$$

$$\beta=\frac{\lambda}{A}(\mu\Omega+E\omega\gamma)\omega\xi_1^2+\frac{\lambda}{D}\Big[\mu\Omega-\frac{\lambda+\mu}{A}(\mu\Omega+E\omega\gamma)\Big]\Big(1-\frac{\lambda+\mu}{A}\omega^2\xi_1^2\Big)\omega\xi_1^2$$

$$\delta=\mu\omega^2\Big[1-\frac{\mu}{A}-\frac{\mu}{D}\omega^2\Big(1-\frac{\lambda+\mu}{A}\Big)^2\xi_1^2\Big]$$

$$\varepsilon=\mu\omega\xi_1\Omega-\frac{\mu}{A}\omega\xi_1(\mu\Omega+E\omega\gamma)-\frac{\mu}{D}\Big[\mu\Omega-\frac{\lambda+\mu}{A}(\mu\Omega+E\omega\gamma)\Big]\Big(1-\frac{\lambda+\mu}{A}\Big)\omega^3\xi_1^3$$

$$\eta=\mu\omega^2\xi_1\Big[\frac{\lambda}{A}+\frac{\lambda}{D}\Big(1-\frac{\lambda+\mu}{A}\omega^2\xi_1^2\Big)\Big(1-\frac{\lambda+\mu}{A}\Big)$$

This problem reduces to solving one ordinary second-order differential equation with variable coefficients and is a computer problem in the general case.

**9. Rods with elongated elliptical sections.** Let $h(\xi_1)=\sqrt{1-\xi_1^2/a^2}$. In place of the variable $\xi_1$ we introduce the variable $x$ by means of the formula $\xi_1=a\cos x,\ 0\leqslant x\leqslant\pi$ and we define the dimensionless displacement and the initial twist by the equalities $u=a\bar u,\ \bar\omega=a\omega$, $\bar\Omega=a\Omega$. Then according to (5.2) and (8.6), the effective stiffnesses are given by the formulas

$$\bar E=2Eba\Big\{\int_0^{\pi}\sin^2x\Big[1-\frac{E}{\bar A}\bar\omega^2\cos^2x-\frac{E}{\bar D}\Big(\frac{\lambda+\mu}{\bar A}\Big)^2\bar\omega^4\cos^4x\Big]dx+\frac{\mu}{E}\inf_{\bar u}\int_0^{\pi}(\bar\alpha\bar u^2_{,x}+\bar\delta\bar u^2-2\bar\varepsilon_1\bar u)\,dx\Big\}\tag{9.1}$$

$$C=2\mu ba\Big\{\int_0^{\pi}\sin^2x\Big[\Big(1-\frac{\mu}{\bar A}\Big)\cos^2x-\frac{\mu}{\bar D}\Big(1-\frac{\lambda+\mu}{\bar A}\Big)^2\bar\omega^2\cos^4x\Big]dx+\inf_{\bar u}\int_0^{\pi}(\bar\alpha\bar u^2_{,x}+\bar\delta\bar u^2-2\bar\varepsilon_2\bar u)\,dx\Big\}\tag{9.2}$$

$$B=2Eba\Big\{\int_0^{\pi}\Big[\sin^2x\Big(-\frac{\mu}{\bar A}\bar\omega\cos^2x+\frac{\mu(\lambda+\mu)}{\bar D\bar A}\Big(1-\frac{\lambda+\mu}{\bar A}\Big)\bar\omega^3\cos^4x\Big)-\frac{\mu}{E}\bar\varepsilon_2\bar u_{(1)}\Big]dx\Big\}$$

$$\bar A=\mu+\Big(E+\frac{\lambda^2}{\lambda+\mu}\Big)\bar\omega^2\cos^2x,\quad \bar D=\lambda+2\mu+\Big[\mu-\frac{(\lambda+\mu)^2}{\bar A}\Big]\bar\omega^2\cos^2x$$

$$\bar\alpha=2+\frac{\lambda}{\mu}-\frac{\lambda^2}{\mu\bar A}\bar\omega^2\cos^2x-\frac{\lambda^2}{\mu\bar D}\Big(1-\frac{\lambda+\mu}{\bar A}\bar\omega^2\cos^2x\Big)^2$$

$$\bar\delta=\bar\omega^2\Big\{\sin^2x\Big[1-\frac{\mu}{\bar A}-\frac{\mu}{\bar D}\Big(1-\frac{\lambda+\mu}{\bar A}\Big)^2\bar\omega^2\cos^2x\Big]-\frac{1}{2}\Big[\sin 2x\Big(\frac{\lambda}{\bar A}+$$

$$\frac{\lambda}{\bar D}\Big(1-\frac{\lambda+\mu}{\bar A}\bar\omega^2\cos^2x\Big)\Big(1-\frac{\lambda+\mu}{\bar A}\Big)\Big)\Big]_{,x}\Big\}$$

$$\bar\varepsilon_1=2(1+\nu)\bar\omega^2\Big\{\Big[\cos^2x\sin x\Big(\frac{\lambda}{\bar A}-\frac{\lambda(\lambda+\mu)}{\bar D\bar A}\Big(1-\frac{\lambda+\mu}{\bar A}\bar\omega^2\cos^2x\Big)\Big)\Big]_{,x}+$$

$$\sin^2x\cos x\Big[-\frac{\mu}{\bar A}-\frac{\mu(\lambda+\mu)}{\bar D\bar A}\Big(1-\frac{\lambda+\mu}{\bar A}\Big)\bar\omega^2\cos^2x\Big]\Big\}$$

$$\bar\varepsilon_2=\bar\omega\Big\{\Big[\cos^2x\sin x\Big(\frac{\lambda}{\bar A}+\frac{\lambda}{\bar D}\Big(1-\frac{\lambda+\mu}{\bar A}\Big)\Big(1-\frac{\lambda+\mu}{\bar A}\bar\omega^2\cos^2x\Big)\Big)\Big]_{,x}+$$

$$\sin^2x\cos x\Big[1-\frac{\mu}{\bar A}-\frac{\mu}{\bar D}\Big(1-\frac{\lambda+\mu}{\bar A}\Big)^2\bar\omega^2\cos^2x\Big]\Big\}$$

The functions $\bar u_{(1)}, \bar u_{(2)}$ that yield the minimum of the functionals (9.1) and (9.2) are found from the solution of the boundary value problems

$$\bar\alpha\bar u_{(1,2),xx}+\bar\alpha_{,x}\bar u_{(1,2),x}-\bar\delta\bar u=-\bar\varepsilon_{(1,2)};\quad \bar u_{(1,2),x}=0\quad\text{for}\quad x=0,\pi\tag{9.3}$$

In the case of small twists, the effective moduli are found analytically

$$\bar E=E\,[\pi ba-{}^1\!/_2(1+\nu)\pi ba\bar\omega^2],\ B=-{}^1\!/_4E\pi ba\bar\omega,$$
$$C={}^1\!/_4\mu(1+\nu)\pi ba\bar\omega^2$$

Within the framework of the asymptotic expression under consideration $b/a\ll 1$ the torsional stiffness $C$ is not in the clssical term $\mu\pi b^3/a$ since it is small asymptotically with respect to that written down (for $\bar\omega\sim 1$). It is seen that the effective Young's modulus decreases as the rod twists while the torsional stiffness increases.

Problems (9.3) were solved by grid factorization. The effective moduli $\bar E, B, C$ were evaluated by numerical integration. The values of the functions $E_*=\bar E/(2Eba),\ B_*=B/(2Eba),\ C_*=C/(2\mu ba)$ are represented in Fig.2 as functions of the parameter $\bar\omega$ for different Poisson's ratios $\nu$, where the solid lines refer to $B_*$, the dashes to $C_*$, and the dash-dot lines to $E_*$. The graphs show that as the twist increases the torsional stiffness grows although $\bar\omega\leqslant 1.1-1.2$, and then starts to decrease. The cross coefficient $B_*$ behaves analogously, while the cross effects

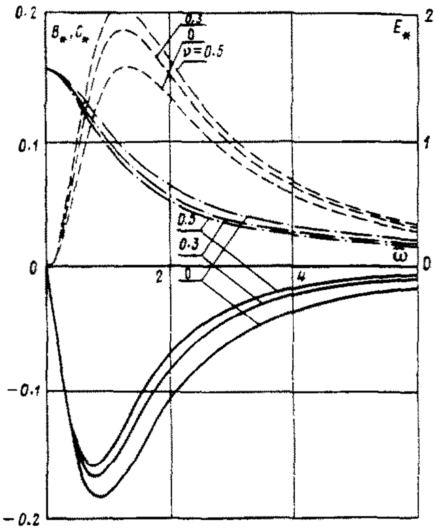



Fig.2

are most noticeable for values of the twist of 0.7–0.8. The effective Young's modulus decreases monotonically as the twist grows.

The presence of the descending branch for the effective coefficients can be explained as follows: for small twists practically the whole section operates at torsion and extension, while for large twists only the "nucleus" of the section operates, that shrinks to the maximal circle inscribed in the cross-section as the twist grows.

## 10. Reduction of the system of equations of the spatial theory of elasticity to systems of two-dimensional equations on a rod section.

Let us formulate a general method for passing from the system of three-dimensional Eqs.(2.1)–(2.4) to systems of equations on sections of a naturally twisted rod. We will seek the solution of problem (2.1)–(2.4) in the form

$$w_\alpha = u_\alpha(\xi) + f_\alpha{}^\sigma Q_\sigma + g_\alpha^\sigma M_\sigma + \theta(\xi) e_{\alpha\beta}\xi^\beta + e_\alpha M + r_\alpha T \tag{10.1}$$
$$w = \psi_\alpha(\xi)\xi^\alpha + f^\sigma Q_\sigma + g^\sigma M_\sigma + u(\xi) + eM + rT, \quad \psi_\alpha = \varphi_\alpha - Du_\alpha$$

Here $T, M$ are the tensile force and torque, $Q_\alpha, M_\sigma$ are components of the transverse force and bending moment vectors for which the change along the axis is given by equilibrium integral Eqs.(3.3)–(3.5), $u(\xi), u_\alpha(\xi)$ are the longitudinal and transverse displacements of the axis, $\theta(\xi)$ is the angle of rotation in the plane of the section, $\psi_\alpha$ are the transverse angles of section rotation including the shear, $f_\alpha{}^\sigma, f^\sigma, g_\alpha{}^\sigma, g^\sigma, e_\alpha, e, r_\alpha, r$ are the desired functions of the coordinate $\xi^\alpha$. Without loss of generality, conditions determining the sense of the kinematic variable $u, u_\alpha, \theta, \psi_\alpha$ can be imposed on the functions in the displacement (10.1):

$$\langle\langle f_\alpha{}^\sigma\rangle\rangle = \langle\langle g_\alpha{}^\sigma\rangle\rangle = \langle\langle e_\alpha\rangle\rangle = \langle\langle r_\alpha\rangle\rangle = \langle\langle f^\sigma\rangle\rangle = \langle\langle g^\sigma\rangle\rangle = \langle\langle e\rangle\rangle = \langle\langle r\rangle\rangle = 0 \tag{10.2}$$

$$e_{\cdot\beta}^{\alpha\cdot}\langle\langle f_\alpha{}^\sigma\xi^\beta\rangle\rangle = e_{\cdot\beta}^{\alpha\cdot}\langle\langle g_\alpha{}^\sigma\xi^\beta\rangle\rangle = e_{\cdot\beta}^{\alpha\cdot}\langle\langle e_\alpha\xi^\beta\rangle\rangle = e_{\cdot\beta}^{\alpha\cdot}\langle\langle r_\alpha\xi^\beta\rangle\rangle = 0 \tag{10.3}$$

$$\langle\langle f^\sigma\xi^\alpha\rangle\rangle = \langle\langle g^\sigma\xi^\alpha\rangle\rangle = \langle\langle e\xi^\alpha\rangle\rangle = \langle\langle r\xi^\alpha\rangle\rangle = 0 \tag{10.4}$$

We express the deformation (2.3) in terms of the displacement (10.1)

$$\varepsilon_{\alpha\beta} = f^\sigma_{(\alpha,\beta)}Q_\sigma + g^\sigma_{(\alpha,\beta)}M_\sigma + e_{(\alpha,\beta)}M + r_{(\alpha,\beta)}T \tag{10.5}$$
$$2\varepsilon_{\alpha3} = \varphi_\alpha - \Omega e_{\alpha\beta}\xi^\beta + (f^\sigma_{,\alpha} + Df_\alpha{}^\sigma + g_\alpha{}^\sigma)Q_\sigma + (g^\sigma_{,\alpha} + Dg_\alpha{}^\sigma)M_\sigma +$$
$$(e_{,\alpha} + De_\alpha)M + (r_{,\alpha} - Dr_\alpha)T$$
$$\varepsilon_{33} = \gamma - \Omega_\alpha\xi^\alpha + (Df^\sigma + g^\sigma)Q_\sigma + M_\sigma Dg^\sigma + MDe + TDr$$
$$\Omega = \theta_{,\xi}, \quad \Omega_\alpha = D\psi_\alpha$$

The quantities $\gamma, \Omega_\alpha, \Omega, \varphi_\alpha$ in (10.5) have the meaning of "rod" measures of the extension of the axis, the bending, the torsion, and the shear. We relate them to the forces and moments by using the compliance matrix

$$\gamma = t^\sigma Q_\sigma + \rho^\sigma M_\sigma + tT + \rho M \tag{10.6}$$
$$\varphi_\alpha = a_\alpha{}^\sigma Q_\sigma + b_\alpha{}^\sigma M_\sigma + h_\alpha T + s_\alpha M$$
$$\Omega_\alpha = c_\alpha{}^\sigma Q_\sigma + d_\alpha{}^\sigma M_\sigma + p_\alpha T + q_\alpha M$$
$$\Omega = i^\sigma Q_\sigma + j^\sigma M_\sigma + iT + jM$$

The coefficients of the compliance matrix are still undetermined constants. We calculate the stress (2.2) by means of the strain (10.5) under the condition (10.6) and we substitute it into system (2.1) and the boundary conditions (2.4). Equating the coefficients of the forces and moments to zero, we obtain a system of two-dimensional equations in the desired functions $f_\alpha{}^\sigma, f^\sigma, g_\alpha{}^\sigma, g^\sigma, e_\alpha, e, r_\alpha, r$:

$$\lambda[(Df^\sigma)_{,\alpha} + g^\sigma_{,\alpha} + f^{\sigma\gamma}_{\gamma,\alpha}] + \mu[2f^{\sigma}_{(\alpha,\beta)}{}^{\beta} + Df^\sigma_{,\alpha} + D^2f_\alpha{}^\sigma + 2Dg_\alpha{}^\sigma + \tag{10.7}$$
$$g^\sigma_{,\alpha}] + \lambda c_\alpha{}^\sigma + \mu[Da_\alpha{}^\sigma + b_\alpha{}^\sigma + e_{\alpha\beta}\xi^\beta(Di^\sigma + j^\sigma)] = 0$$
$$\mu[f^{\sigma\alpha}_{,\alpha} + (Df_\alpha{}^\sigma)_{,}{}^{\alpha} + g^{\sigma\alpha}_{\alpha,}] + (\lambda + 2\mu)(D^2f^\sigma + 2Dg^\sigma) + \lambda(Df^{\sigma\alpha}_{\alpha,} + g^{\sigma\alpha}_{\alpha,}) +$$
$$(\lambda + 2\mu)[Dt^\sigma + \rho^\sigma + \xi^\alpha(Dc_\alpha{}^\sigma + d_\alpha{}^\sigma)] = 0$$

$$\lambda[(Dg^\sigma)_{,\alpha} + g^{\sigma\gamma}_{\gamma,\alpha}] + \mu(2g^{\sigma}_{(\alpha,\beta)}{}^{\beta} + Dg^\sigma_{,\alpha} + D^2g_\alpha{}^\sigma) + \lambda d_\alpha{}^\sigma + \tag{10.8}$$
$$+ \mu(Db_\alpha{}^\sigma + e_{\alpha\beta}\xi^\beta Dj^\sigma) = 0$$

$$\mu[g^{\sigma\alpha}_{,\alpha} + (Dg_\alpha{}^\sigma)_,{}^\alpha] + (\lambda + 2\mu) D^2 g^\sigma + \lambda D g^{\sigma\alpha}_{\alpha,} + (\lambda + 2\mu) \times (D\rho^\sigma + \xi^\alpha D d_\alpha{}^\sigma) = 0$$

$$\lambda[(De)_{,\alpha} + e_{\gamma,\alpha}^{\ \gamma}] + \mu(2e_{(\alpha,\beta)}^{\ \beta} + De_{,\alpha} + D^2 e_\alpha) + \lambda g_\alpha + \mu D s_\alpha = 0 \quad (10.9)$$

$$\mu[e^\alpha_{,\alpha} + (De_\alpha)_,{}^\alpha] + (\lambda + 2\mu) D^2 e + \lambda D e_{\alpha,}{}^\alpha + (\lambda + 2\mu)\xi^\alpha D g_\alpha = 0$$

$$\lambda[(Dr)_{,\alpha} + r_{\gamma,\alpha}^{\ \gamma}] + \mu(2r_{(\alpha,\beta)}^{\ \beta} + Dr_{,\alpha} + D^2 r_\alpha) + \lambda p_\alpha + \mu D h_\alpha = 0 \quad (10.10)$$

$$\mu[r^\alpha_{,\alpha} + (Dr_\alpha)_,{}^\alpha] + (\lambda + 2\mu) D^2 r + \lambda D r_{\alpha,}{}^\alpha + (\lambda + 2\mu)\xi^\alpha D p_\alpha = 0$$

and a system of Neumann-type boundary conditions

$$[\lambda(Df^\sigma + g^\sigma + f^{\sigma\gamma}_{\gamma,})\delta_{\alpha\beta} + \mu(2f^\sigma_{(\alpha,\beta)} - \omega e_{\nu\beta}\xi^\nu(f^\sigma_{,\alpha} + Df_\alpha{}^\sigma + g_\alpha{}^\sigma)) + \quad (10.11)$$
$$\lambda(t^\sigma + c_\nu{}^\sigma\xi^\nu)\delta_{\alpha\beta} - \mu\omega e_{\nu\beta}\xi^\nu(a_\alpha{}^\sigma + e_{\alpha\gamma}\xi^\gamma i^\sigma)]\nu^\beta = 0$$

$$[\mu(f^\sigma_{,\alpha} + Df_\alpha{}^\sigma + g_\alpha{}^\sigma) - \omega e_{\rho\alpha}\xi^\rho((\lambda + 2\mu)(Df^\sigma + g^\sigma) + \lambda f^{\sigma\gamma}_{,\gamma}) +$$
$$\mu(a_\alpha{}^\sigma + e_{\alpha\beta}\xi^\beta i^\sigma) - (\lambda + 2\mu)\omega e_{\rho\alpha}\xi^\rho(t^\sigma + \xi^\gamma c_\gamma{}^\sigma)]\nu^\alpha = 0$$

$$[\lambda(Dg^\sigma + g^{\sigma\gamma}_{\gamma,})\delta_{\alpha\beta} + \mu(2g^\sigma_{(\alpha,\beta)} - \omega e_{\nu\beta}\xi^\nu(g^\sigma_{,\alpha} + Dg_\alpha{}^\sigma)) + \quad (10.12)$$
$$\lambda(\rho^\sigma + d_\nu{}^\sigma\xi^\nu)\delta_{\alpha\beta} - \mu\omega e_{\nu\beta}\xi^\nu(b_\alpha{}^\sigma + e_{\alpha\gamma}\xi^\gamma j^\sigma)]\nu^\beta = 0$$

$$[\mu(g^\sigma_{,\alpha} + Dg_\alpha{}^\sigma) - \omega e_{\rho\alpha}\xi^\rho((\lambda + 2\mu)Dg^\sigma + \lambda g^{\sigma\gamma}_{\gamma,}) + \mu(b_\alpha{}^\sigma + e_{\alpha\beta}\xi^\beta j^\sigma) -$$
$$(\lambda + 2\mu)\omega e_{\rho\alpha}\xi^\rho(\rho^\sigma + \xi^\gamma d_\gamma{}^\sigma)]\nu^\alpha = 0$$

$$[\lambda(De + e^{\gamma}_{\gamma,})\delta_{\alpha\beta} + \mu(2e_{(\alpha,\beta)} - \omega e_{\nu\beta}\xi^\nu(e_{,\alpha} + De_\alpha)) + \quad (10.13)$$
$$\lambda(\rho + g_\gamma\xi^\gamma)\delta_{\alpha\beta} - \mu\omega e_{\nu\beta}\xi^\nu(s_\alpha + e_{\alpha\gamma}\xi^\gamma j)]\nu^\beta = 0$$

$$[\mu(e_{,\alpha} + De_\alpha) - \omega e_{\nu\alpha}\xi^\nu((\lambda + 2\mu)De + \lambda e^{\gamma}_{\gamma,}) + \mu(s_\alpha + e_{\alpha\beta}\xi^\beta j) -$$
$$(\lambda + 2\mu)\omega e_{\sigma\alpha}\xi^\sigma(\rho + \xi^\gamma g_\gamma)]\nu^\alpha = 0$$

$$[\lambda(Dr + r^{\gamma}_{\gamma,})\delta_{\alpha\beta} + \mu(2r_{(\alpha,\beta)} - \omega e_{\nu\beta}\xi^\nu(r_{,\alpha} + Dr_\alpha)) + \quad (10.14)$$
$$\lambda(t + p_\nu\xi^\nu)\delta_{\alpha\beta} - \mu\omega e_{\nu\beta}\xi^\nu(h_\alpha + e_{\alpha\gamma}\xi^\gamma i)]\nu^\beta = 0$$

$$[\mu(r_{,\alpha} + Dr_\alpha) - \omega e_{\nu\alpha}\xi^\nu((\lambda + 2\mu)Dr + \lambda r^{\gamma}_{\gamma,}) + \mu(h_\alpha + e_{\alpha\beta}\xi^\beta i) -$$
$$(\lambda + 2\mu)\omega e_{\sigma\alpha}\xi^\sigma(t + \xi^\gamma p_\gamma)]\nu^\alpha = 0$$

Problem (10.7)–(10.14) consists of six Eqs.(10.8) with boundary conditions (10.12) in the six functions $g^\tau$, $g_\gamma{}^\tau$, six Eqs.(10.7) with the boundary conditions (10.11) in the 12 desired functions $g^\tau$, $g_\gamma{}^\tau$, $f^\tau$, $f_\gamma{}^\tau$ and two systems of identical structure (10.9), (10.10) with the boundary conditions (10.13), (10.14) in the three desired functions $e$, $e_\gamma$ and $r$, $r_\gamma$, respectively in each. Under the constraints (10.2)–(10.4), problem (10.7)–(10.14) with the addition of the equation of state (3.1) is solvable single-valuedly.

The functions $f_\gamma{}^\tau$, $g_\gamma{}^\tau$, $f^\tau$, $g^\tau$ and $e_\alpha$, $e$, $r_\alpha$, $r$ respectively characterize the bending and stretching-torsion of the rod. In the general case, the solution of the bending problem is combined with the problem of longitudinally torsional deformation in terms of the coefficients of the compliance matrix. If the axis of the natural twist $x^3$ coincides with the axis of the centres of gravity of the transverse sections, and the section is centrally-symmetric about this axis, it can be confirmed that the problem of rod deformation decomposes into two independent problems, one of which describes the bending and the other the longitudinally torsional deformations.

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